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Dependent coordinates in path integral measure factorization

S N Storchak

Institute for High Energy Physics, Protvino, Moscow Region, 142284, Russia

E-mail: storchak@mx.ihep.su

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Abstract
The transformation of the path integral measure under the reduction procedure in dynamical systems with a symmetry is considered. Investigation is carried out in the case of the Wiener-type path integrals that are used for description of the diffusion on a smooth compact Riemannian manifold with the given free isometric action of the compact semisimple unimodular Lie group. The transformation of the path integral, which factorizes the path integral measure, is based on the application of the optimal nonlinear filtering equation from the stochastic theory. The integral relation between the kernels of the original and reduced semigroups is obtained.

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1. Introduction

The standard approach to path integral quantization of gauge field theories is based on the Faddeev–Popov method [1] by which a path integral over invariant variables is rewritten as a path integral over variables constrained by some gauge conditions. But in order to obtain such a representation, it is necessary to separate the path integral measure into two parts related, correspondingly, to gauge-invariant (independent) and gauge-dependent (or group) degrees of freedom. However, in general it is unknown whether or not this separation of the path integral measure leads to some Jacobian in the path integral measure.

In the present paper we consider the path integral quantization problem for a scalar particle which moves on a smooth compact Riemannian manifold (without boundary). We suppose that there exists a free effective isometric action of the compact semisimple unimodular Lie group on this manifold. In the problem considered, as in gauge field theories, an original manifold can be regarded as a total space of the principal fibre bundle. The principal bundle picture permits us to introduce new coordinates on the original manifold. As the coordinate functions of a chart in the manifold atlas, we can take the invariant variables and the group-valued variables. The invariant variables are related to the orbit space of the fibre bundle. And the group-valued variables are defined by the group element which ‘measures’ the distance between a point considered in the total space of the fibre bundle and a point in the base space which can be reached along the orbit.

The path integral quantization of the present problem was considered in [2, 3] by using the definitions of the path integrals based on discrete approximations. In our papers [4, 5], we studied the quantization of this problem with the help of the path integrals in which the path integral measures were defined by stochastic processes. It was found there that the factorization of the path integral measure can be performed by applying the nonlinear filtering equation from the stochastic process theory.

In gauge field theories the description of the evolution on the orbit space is usually given by imposing the gauge conditions (or gauges) on the gauge variables. From the local point of view the gauge conditions (given by the system of equations) define a local submanifold in the original manifold. In turn, the submanifold can be viewed as a local section of a principal fibre bundle.

There is a local isomorphism between the initial principal fibre bundle and the trivial principal fibre bundle which has this local submanifold as the base space [6]. It allows us to use the constrained (or dependent) coordinates as the coordinates on our principal fibre bundle.

If these local submanifolds (local sections) are parts of some global submanifold (a global section), then our initial principal fibre bundle is a trivial one. And in this case, our dependent coordinates have a global meaning. But in general, there is no global section in the principal fibre bundle. So, in a local neighbourhood of each point of the initial manifold, one should introduce its own dependent coordinates.

After performing local factorization of the path integral measure into the ‘group measure’ and the measure that is given on the local section, one should solve the problem of the definition of a global measure related to the set of these local measures.

A possible solution of such a problem was proposed in [7], where the global path integral measure was defined in terms of local measures given on the local sections. Therefore, we will study the dependent coordinate description of particle motion in the particular case of the trivial principal fibre bundle.

The representation of the orbit space path integral as the integral over the dependent variables was studied in many papers (see, for example, [8–10]). But the path integral measure factorization questions were not considered there. In the present paper, we will investigate the transformation of the path integral measure on changing the path integral variables for the dependent ones in the Wiener-like path integrals that are used to describe the ‘quantum’ motion of a scalar particle on a manifold.

2. Definitions

The objects of our consideration will be the path integrals representing the solution of the backward Kolmogorov equation given on a smooth compact Riemannian manifold $\mathcal{P}$:

$$\begin{cases} \left(\dfrac{\partial}{\partial t_a}+\dfrac{1}{2}\mu^2\kappa\triangle_{\mathcal{P}}(p_a)+\dfrac{1}{\mu^2\kappa m}V(p_a)\right)\psi_{t_b}(p_a,t_a)=0 \\ \psi_{t_b}(p_b,t_b)=\phi_0(p_b) \qquad (t_b>t_a). \end{cases} \tag{1}$$

In this equation $\mu^2=\frac{\hbar}{m}$, κ is a real positive parameter, $\triangle_{\mathcal{P}}(p_a)$ is a Laplace–Beltrami operator on manifold $\mathcal{P}$ and $V(p)$ is a group-invariant potential term. In a chart $(\mathcal{U},\varphi^A)$ with the coordinate functions $Q^A=\varphi^A(p)$, $p\in\mathcal{P}$, the Laplace–Beltrami operator is given as

$$\triangle_{\mathcal{P}}(Q) = G^{-1/2}(Q)\frac{\partial}{\partial Q^A}G^{AB}(Q)G^{1/2}(Q)\frac{\partial}{\partial Q^B}$$

where the matrix $G^{AB}(Q)$ is the inverse of the matrix of the Riemannian metric G_{AB} given in the coordinate basis $\left\{\frac{\partial}{\partial Q^A}\right\}$; $G = \det(G_{AB})$.

In order to obtain the Schrödinger equation from equation (1), one should perform a transition to the forward Kolmogorov equation and then change κ for i. If the coefficients of the backward Kolmogorov equation are properly restricted to the definite class of functions, then the fundamental solution of a backward equation also satisfies a forward equation. However, the transition from the Wiener-like path integrals to the Feynman ones is a special problem which needs further investigation and is not considered in the paper.

There are different representations of the solution of equation (1) in terms of the path integral. We will use the definition of the path integral in [11], together with the assumptions that all necessary smoothness conditions take place in our paper. From [11], the solution of (1) can be written as follows:

$$\begin{aligned}\psi_{t_b}(p_a, t_a) &= \mathrm{E}\left[\phi_0(\eta(t_b))\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b} V(\eta(u))\,\mathrm{d}u\right\}\right]\\ &= \int_{\Omega_-}\mathrm{d}\mu^{\eta}(\omega)\phi_0(\eta(t_b))\exp\{\cdots\}\end{aligned}\tag{2}$$

where $\eta(t)$ is a global stochastic process on a manifold $\mathcal{P}$. $\Omega_- = \{\omega(t) : \omega(t_a) = 0, \eta(t) = p_a + \omega(t)\}$ is the path space on this manifold and the path integral measure μ^{η} is defined by the probability distribution of a stochastic process $\eta(t)$.

In the chart $(\mathcal{U}, \varphi)$ from the atlas of the manifold $\mathcal{P}$, the global stochastic process $\eta(t)$ is given by the local processes $\varphi(\eta) = \eta_{\varphi}(t) \equiv \{\eta^A(t)\}$ that are solutions of the stochastic differential equations:

$$\mathrm{d}\eta^A(t) = \frac{1}{2}\mu^2\kappa G^{-1/2}\frac{\partial}{\partial Q^B}(G^{1/2}G^{AB})\,\mathrm{d}t + \mu\sqrt{\kappa}\mathfrak{X}^A_{\bar{M}}(\eta(t))\,\mathrm{d}w^{\bar{M}}(t)\tag{3}$$

where the matrix $\mathfrak{X}^A_{\bar{M}}$ is defined by the local equality $\sum_{\bar{K}=1}^{n_P}\mathfrak{X}^A_{\bar{K}}\mathfrak{X}^B_{\bar{K}} = G^{AB}$.
(Here and in what follows we denote the Euclidean indices by overbarred indices.)

According to [11], the global semigroup determined by equation (2) acts in the space of smooth and bounded functions on $\mathcal{P}$. It is defined by the limit (under the refinement of the time interval) of the superposition of the local semigroups

$$\psi_{t_b}(p_a, t_a) = U(t_b, t_a)\phi_0(p_a) = \lim\nolimits_q \tilde{U}_{\eta}(t_a, t_1)\cdots\tilde{U}_{\eta}(t_{n-1}, t_b)\phi_0(p_a)\tag{4}$$

where, in turn, each of the local semigroups $\tilde{U}_{\eta}$ is related to the local representative of the global stochastic process η.

One of the main advantages of definition (4) is that it permits us to derive the transformation properties of the path integral (2) by studying the local semigroups $\tilde{U}_{\eta}$:[1]

$$\tilde{U}_{\eta}(s, t)\phi(p) = \mathrm{E}_{s,p}\phi(\eta(t)) \qquad s \leqslant t \quad \eta(s) = p.$$

These local semigroups are also given by the path integrals with the integration measures determined by the local representatives $\varphi^{\mathcal{P}}(\eta(t)) = \eta^{\varphi^{\mathcal{P}}}(t) \equiv \{\eta^A(t)\}$ of the global stochastic process $\eta(t)$.

[1] In the following, we omit the potential term of the Hamiltonian operator as it is inessential for us in performing the path integral transformations. It will be recovered in our final formulae.

3. Principal fibre bundle coordinates

The problem, which we consider in the paper, is related to the investigation of the reduction procedure in dynamical systems with symmetry. Due to a symmetry, the initial dynamical system is reduced to a system that can be described in terms of some invariant variables. The geometry of this problem is well developed [12].

A free (effective) action of the compact semisimple group Lie $\mathcal{G}$ on a smooth compact manifold[2] $\mathcal{P}$ leads to the orbit-fibreing of the manifold $\mathcal{P}$. And we can regard the manifold $\mathcal{P}$ as the total space of the principal fibre bundle $P(\mathcal{M},\mathcal{G})$ where the orbit space $\mathcal{M}$ is a base space.

From the bundle picture it follows that locally manifold $\mathcal{P}$ has the product structure: $\pi^{-1}(\mathcal{U}_x) \sim \mathcal{U}_x \times \mathcal{G}$, where $\mathcal{U}_x$ is an open neighbourhood of the point $x = \pi(p)$ which belongs to the chart $(\mathcal{U}_x, \varphi_x)$ of the bundle $P(\mathcal{M},\mathcal{G})$. Therefore, we can use the principal bundle coordinates (x^i, a^α) $(i = 1,\ldots,N_{\mathcal{M}}, N_{\mathcal{M}} = \dim\mathcal{M}, \alpha = 1,\ldots,N_{\mathcal{G}}, N_{\mathcal{G}} = \dim\mathcal{G}$, $N_{\mathcal{P}} = N_{\mathcal{M}} + N_{\mathcal{G}})$ of the point $p \in \mathcal{P}$ instead of the initial coordinates Q^A of this point. The problem consists of a definition of the coordinates (x^i, a^α) from the known initial coordinates Q^A.

As invariant coordinates $x^i(Q)$ (the orbit space coordinates) of a point p, it is possible to take a set of functionally independent and $\mathcal{G}$-invariant functions that are solutions of the special differential equations. In many cases, it is very difficult to find the solutions of these differential equations. However, there exists another method of the orbit space coordinatization in which the necessary invariant coordinates are defined with the help of the gauge constraints.

It is supposed that in each sufficiently small neighbourhood of the point $p \in \mathcal{P}$, there is a set of functions $\{\chi^\alpha(Q), \alpha = 1,\ldots,N_{\mathcal{G}}\}$ that can be used (by the equation $\chi^\alpha(Q) = 0$) to determine a local submanifold of the manifold $\mathcal{P}$. (This submanifold should have the transversal intersection with each of the orbits.) Then the coordinates on the manifold $\mathcal{P}$ can be introduced as follows.

By our assumption, we have the action of the group $\mathcal{G}$ on the manifold $\mathcal{P}$: $\tilde{p} = pa$, or in coordinates: $\tilde{Q}^A = F^A(Q^B, a^\alpha)$, where Q^A are the coordinates of a point p. We assume that it is a right action, i.e. $(pa_1)a_2 = p(a_1a_2)$:

$$F(F(Q,a_1),a_2) = F(Q,\Phi(a_1,a_2))$$

where Φ is the function which determines the group multiplication law in the space of the group parameters.

The group coordinates $a^\alpha(Q)$ of a point $p \in \mathcal{P}$ are defined as coordinates of that group element which carries the point p to the local submanifold $\{\chi^\alpha(Q) = 0\}$. These group coordinates are given by the solution of the following equation:

$$\chi^\alpha(F^A(Q, a^{-1}(Q))) = 0.$$

The invariant coordinates $x^i(Q)$ of a point p are the coordinates of that point of the submanifold $\{\chi^\alpha(Q) = 0\}$ which is obtained from the point p under the action of the group element with the coordinates $a^\alpha(Q)$. If the submanifold $\{\chi^\alpha(Q) = 0\}$ is given parametrically: $Q^A = Q^{*A}(x^i)$, the coordinates $x^i(Q)$ are defined by the equation

$$Q^{*A}(x^i) = F^A(Q, a^{-1}(Q)).$$

Such an approach to the coordinatization of the orbit space was considered in [13]. Besides, in these papers the geometrical generalization of the Bogolubov coordinate transformation method from [14] was given.

[2] In our case this is an isometric action on a Riemannian manifold.

The path integral transformation, induced by the replacement of the coordinates Q^A for the coordinates $(x^i(Q), a^\alpha(Q))$ and the factorization of the path integral measure, was investigated in [5]. In the present paper we study the same problem, but now together with the group coordinates a^α (obtained by a previous method) we will use the constrained (or dependent) coordinates Q^{*A}: $\{\chi^\alpha(Q^{*A}) = 0\}$. The point with the coordinates Q^{*A} belongs to the local submanifold $\{\chi^\alpha = 0\}$ in the original manifold $\mathcal{P}$.

We will assume that these local submanifolds form a global submanifold Σ in the original manifold $\mathcal{P}$. Therefore, our principal fibre bundle $P(\mathcal{M}, \mathcal{G})$ is a trivial one.

Notice that there is a local isomorphism of the principal fibre bundle $P(\mathcal{M}, \mathcal{G})$ and the trivial principal fibre bundle $P_\Sigma = \Sigma \times \mathcal{G} \to \Sigma$ [6, 7]. From this fact it follows that for the total space $\mathcal{P}$ of the principal fibre bundle $P(\mathcal{M}, \mathcal{G})$, we can define a new atlas with the charts that are related to the submanifold Σ.

The relation between the constrained coordinates Q^* and the initial coordinates Q^A of a point p is given by the equation $Q^A = F^A(Q^{*A}, a^\alpha)$. Later we will see that an apparent ambiguity of a transition from Q^A to (Q^{*A}, a^α) is compensated by the presence of the corresponding projection operators in the resulting expressions.

The representation of a Riemannian metric of the manifold $\mathcal{P}$ in new coordinates (Q^{*A}, a^α) is obtained from the transformation of the coordinate vector fields. It is given as follows:

$$\frac{\partial}{\partial Q^B} = F_B^C(F(Q^*, a), a^{-1}) N_C^A(Q^*) \frac{\partial}{\partial Q^{*A}} + F_B^E(F(Q^*, a), a^{-1}) \chi_E^\mu(Q^*) (\Phi^{-1})_\mu^\beta(Q^*) \bar{v}_\beta^\alpha(a) \frac{\partial}{\partial a^\alpha}. \tag{5}$$

Here $F_B^C(Q, a) \equiv \frac{\partial F^C}{\partial Q^B}(Q, a)$, $\chi_E^\mu \equiv \frac{\partial \chi^\mu}{\partial Q^E}(Q)$, $(\Phi^{-1})_\mu^\beta(Q)$—the matrix which is inverse to the Faddeev–Popov matrix:

$$(\Phi)_\mu^\beta(Q) = K_\mu^A(Q) \frac{\partial \chi^\beta(Q)}{\partial Q^A}$$

(K_μ are the Killing vector fields for the Riemannian metric $G_{AB}(Q)$), the matrix $\bar{v}_\beta^\alpha(a)$ is the inverse of the matrix $\bar{u}_\beta^\alpha(a)$. $\det \bar{u}_\beta^\alpha(a)$ is the density of a right-invariant measure given on the group $\mathcal{G}$.

N_C^A is the projection operator $\left(N_B^A N_C^B = N_C^A\right)$ onto the subspace which is orthogonal to the Killing vector field subspace:

$$N_C^A(Q) = \delta_C^A - K_\alpha^A(Q) (\Phi^{-1})_\mu^\alpha(Q) \chi_C^\mu(Q).$$

In (5), this projection operator is restricted to the submanifold $\{\chi^\alpha = 0\}$:

$$N_D^M(Q^*) \equiv N_D^M(F(Q^*, e)) \qquad N_D^M(Q^*) = F_D^B(Q^*, a) N_B^A(F(Q^*, a)) F_A^M(F(Q^*, a), a^{-1})$$

e is the unity element of the group. Formula (5) is similar to the corresponding formula from [9, 15]. Many facts about the dependent coordinate can be found, for example, in [16].

As an operator, the vector field $\frac{\partial}{\partial Q^{*A}}$ is defined by the rule

$$\frac{\partial}{\partial Q^{*A}} \varphi(Q^*) = (P_\perp)_A^D(Q^*) \left.\frac{\partial \varphi(Q)}{\partial Q^D}\right|_{Q=Q^*}$$

where $P_\perp$ is a projection operator on the tangent plane to the submanifold given by the gauges χ:

$$(P_\perp)_B^A = \delta_B^A - \chi_B^\alpha (\chi\chi^\top)^{-1}{}_\alpha^\beta (\chi^\top)_\beta^A.$$

Here $(\chi^\top)_\beta^A$ is a transposed matrix to the matrix χ_B^ν:

$$(\chi^\top)_\mu^A = G^{AB} \gamma_{\mu\nu} \chi_B^\nu \qquad \gamma_{\mu\nu} = K_\mu^A G_{AB} K_\nu^B.$$

The above projection operators have the following properties:

$$(P_\perp)^{\tilde{A}}_B N^C_{\tilde{A}} = (P_\perp)^C_B \qquad N^{\tilde{A}}_B (P_\perp)^C_{\tilde{A}} = N^C_B.$$

In the new coordinate basis the metric G_{AB} is written as a metric $\tilde{G}_{\mathcal{AB}}(Q^*, a)$ with the following components:

$$\begin{pmatrix} G_{CD}(Q^*)(P_\perp)^C_A (P_\perp)^D_B & G_{CD}(Q^*)(P_\perp)^D_A K^C_\mu \bar{u}^\mu_\alpha(a) \\ G_{CD}(Q^*)(P_\perp)^C_A K^D_\nu \bar{u}^\nu_\beta(a) & \gamma_{\mu\nu}(Q^*)\bar{u}^\mu_\alpha(a)\bar{u}^\nu_\beta(a) \end{pmatrix} \tag{6}$$

where the projection operators $P_\perp$ depend on Q^*, i.e., they are restricted to the submanifold, $G_{CD}(Q^*) \equiv G_{CD}(F(Q^*, e))$:

$$G_{CD}(Q^*) = F^M_C(Q^*, a) F^N_D(Q^*, a) G_{MN}(F(Q^*, a)).$$

The pseudoinverse matrix $\tilde{G}^{\mathcal{AB}}(Q^*, a)$ to matrix (6) is as follows:

$$\begin{pmatrix} G^{EF} N^C_E N^D_F & G^{SD} N^C_S \chi^\mu_D (\Phi^{-1})^\nu_\mu \bar{v}^\sigma_\nu \\ G^{CB}\chi^\gamma_C (\Phi^{-1})^\beta_\gamma N^D_B \bar{v}^\alpha_\beta & G^{CB}\chi^\gamma_C (\Phi^{-1})^\beta_\gamma \chi^\mu_B (\Phi^{-1})^\nu_\mu \bar{v}^\alpha_\beta \bar{v}^\sigma_\nu \end{pmatrix}. \tag{7}$$

In (7), $\bar{v}^\sigma_\nu \equiv \bar{v}^\sigma_\nu(a)$ and other components depend on Q^*.

The pseudoinversion of $\tilde{G}_{\mathcal{BC}}$ means that

$$\tilde{G}^{\mathcal{AB}}\tilde{G}_{\mathcal{BC}} = \begin{pmatrix} (P_\perp)^C_B & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix}.$$

The determinant of matrix (6) is equal to

$$(\det \tilde{G}_{\mathcal{AB}}) = \det G_{AB}(Q^*) \det \gamma_{\alpha\beta}(Q^*) (\det \chi\chi^\top)^{-1}(Q^*) \big(\det \bar{u}^\mu_\nu(a)\big)^2 \times \big(\det \Phi^\alpha_\beta(Q^*)\big)^2 \det(P_\perp)^C_B(Q^*).$$

It does not vanish only on the surface $\{\chi = 0\}$. On this surface $\det(P_\perp)^C_B$ is equal to unity.

4. Transformation of the stochastic process and the semigroup

The replacement of the coordinates leads to a new representation of the local stochastic process $\eta^A(t)$ on the principal fibre bundle. From the obtained local processes, we can form, by using the method in [11], a new global process $\zeta(t)$. This means that we have performed the transformation of the global process $\eta(t)$ to the process $\zeta(t)$.

The global process $\zeta(t)$ has two kinds of local components: $(Q^{*A}(t), a^\alpha(t))$. The components $a^\alpha(t)$ describe the part of the stochastic evolution that originates from the stochastic evolution that was given on the group $\mathcal{G}$. The $Q^{*A}(t)$ evolution has its origin in the stochastic evolution given on the submanifold Σ.

Although the process $Q^{*A}(t)$ is described in terms of the dependent coordinates, the transformation of the local stochastic process $\eta^A(t)$ to the local process $\zeta^{\mathcal{A}}(t) = (Q^{*A}(t), a^\alpha(t))$ is, in fact, the phase space transformation of the process $\eta^A(t)$. It takes place since the condition $\chi^\alpha(Q^*) = 0$, which is also valid for the stochastic processes $Q^*(t)$, restricts a number of dependent coordinates.

But it is known that the phase space transformation of the stochastic processes does not change the probabilities. It means that the action of the local semigroup $\tilde{U}_\eta$ on a function $\varphi_0(p)$ is equal to the expectation of the transformed function given a σ algebra generated by the transformed process $\zeta^{\mathcal{A}}(t)$.

In charts of the manifold $\mathcal{P}$, this transition to new coordinates can be considered as follows. The local semigroup

$$\tilde{U}_\eta(s,t)\phi_0(p) = \mathrm{E}_{s,p}\phi_0(\eta(t)) \qquad s \leqslant t \quad \eta(s) = p$$

for the process $\eta(t)$ which is restricted to the chart $(\mathcal{V}_p, \varphi^{\mathcal{P}})$ with

$$\varphi^{\mathcal{P}}(\eta(t)) = \eta^{\varphi^{\mathcal{P}}}(t) \equiv \{\eta^A(t)\}$$

can be written as

$$\tilde{U}_\eta(s,t)\phi_0(p) = \mathrm{E}_{s,\varphi^{\mathcal{P}}(p)}\phi_0\big((\varphi^{\mathcal{P}})^{-1}\big(\eta^{\varphi^{\mathcal{P}}}(t)\big)\big) \qquad \eta^{\varphi^{\mathcal{P}}}(s) = \varphi^{\mathcal{P}}(p).$$

The phase space transformation of the local stochastic processes

$$\eta^A(t) = F^A(Q^{*B}(t), a^\alpha(t))$$

transforms the local semigroup $\tilde{U}_\eta$:

$$\tilde{U}_\eta(s,t)\phi_0(p) = \mathrm{E}_{s,\tilde{\varphi}^{\mathcal{P}}(p)}\phi_0\big((\tilde{\varphi}^{\mathcal{P}})^{-1}\big(\zeta^{\tilde{\varphi}^{\mathcal{P}}}(t)\big)\big) = \mathrm{E}_{s,\tilde{\varphi}^{\mathcal{P}}(p)}\tilde{\phi}_0\big(\zeta^{\tilde{\varphi}^{\mathcal{P}}}(t)\big)$$

where $(\tilde{\varphi}^{\mathcal{P}})^{-1} = (\varphi^{\mathcal{P}})^{-1} \circ F$ and $\tilde{\phi}_0 = \phi_0 \circ (\tilde{\varphi}^{\mathcal{P}})^{-1}$.

Therefore, after changing the coordinates in our local semigroups we should take the expectation values with respect to the measures defined by the probability distribution of the local processes $\zeta^{\tilde{\varphi}^{\mathcal{P}}}(t) = (Q^{*A}(t), a^\alpha(t))$. If these processes are consistent with each other on overlapping of the charts, we can define, by the method in [11], the global process and global semigroup. In turn, the consistence of the local processes is verified by studying the transformations of the local stochastic differential equations that are used to define the local stochastic processes.

5. Stochastic differential equations

Let us consider the stochastic differential equation for the component Q^{*A} of the local stochastic process $\zeta^{\mathcal{A}}(t) = (Q^{*A}(t), a^\alpha(t))$. We assume that the stochastic differential equation for this variable has the following form:

$$\mathrm{d}Q^{*A}(t) = b^{*A}(t)\,\mathrm{d}t + c^{*A}_{\bar{B}}(t)\,\mathrm{d}w^{\bar{B}}(t) \tag{8}$$

where we should define explicitly the drift and the diffusion coefficients.

We know that the variable Q^{*A} is related to the variable Q^A by the equation:

$$Q^{*A} = F^A(Q, a^{-1}(Q)).$$

There exists the same relation between the stochastic variable $Q^{*A}(t)$ and the stochastic variable $\eta^A(t)$.

Therefore, we can apply the Itô differentiation formula to the stochastic variable $Q^{*A}(t)$. We get

$$\mathrm{d}Q^{*A}(t) = \frac{\partial Q^{*A}}{\partial Q^E}\,\mathrm{d}\eta^E(t) + \frac{1}{2}\frac{\partial^2 Q^{*A}}{\partial Q^E \partial Q^C}\langle \mathrm{d}\eta^E(t)\,\mathrm{d}\eta^C(t)\rangle. \tag{9}$$

Then, putting the expression of the stochastic differential $\mathrm{d}\eta^A(t)$ from (3) into the right-hand side of (9), we obtain:

$$\begin{aligned}\mathrm{d}Q^{*A}(t) = {}& \frac{\partial Q^{*A}}{\partial Q^E}\left(-\frac{1}{2}\mu^2\kappa G^{PB}(\eta(t))\Gamma^E_{PB}(\eta(t))\,\mathrm{d}t + \mu\sqrt{\kappa}\mathfrak{X}^E_{\bar{M}}(\eta(t))\,\mathrm{d}w^{\bar{M}}(t)\right)\\ &+ \frac{1}{2}\frac{\partial^2 Q^{*A}}{\partial Q^E \partial Q^C}\langle \mathrm{d}\eta^E(t)\,\mathrm{d}\eta^C(t)\rangle\end{aligned} \tag{10}$$

where Γ^E_{PB} are the Christoffel coefficients for the Riemannian metric G_{AB}.

In order to obtain the stochastic differential equation for $Q^*(t)$, we should express the stochastic variable $\eta^A(t)$ in the last equation in terms of $Q^{*A}(t)$ and $a^\alpha(t)$. It can be done by using the equation $\eta^A(t) = F^A(Q^{*B}(t), a^\alpha(t))$.

Performing this transformation, we find that the coefficient differential $\mathrm{d}t$ in the obtained expression will be the drift of equation (8). And, correspondingly, the term with the stochastic differential $\mathrm{d}w(t)$ will be the diffusion coefficient. As a result, we obtain the following equation for $Q^*(t)$:

$$\begin{aligned}\mathrm{d}Q^{*A}(t) = \frac{1}{2}\mu^2\kappa\bigg[&N^A_C N^R_M G(Q^*(t))^{-1/2}\frac{\partial}{\partial Q^{*R}}(G(Q^*(t))^{1/2}G^{CM}(Q^*(t)))\\ &+N^A_{CL}G^{CL} - G^{PC}N^K_C K^M_{\mu M}(\Phi^{-1})^\mu_\nu\chi^\nu_P + G^{PC}N^A_C K^E_{\mu P}(\Phi^{-1})^\mu_\nu\chi^\nu_E\\ &+G^{PB}N^A_C K^C_{\mu B}(\Phi^{-1})^\mu_\nu\chi^\nu_P\bigg]\mathrm{d}t + \mu\sqrt{\kappa}N^A_C\tilde{\mathfrak{X}}^C_{\bar{M}}(Q^*(t))\,\mathrm{d}w^{\bar{M}}(t).\end{aligned} \tag{11}$$

In this equation all variables depend on $Q^*(t)$ and by additional lower indices we denote the corresponding derivatives. (For example, $N^A_{CL}(Q^*) \equiv \frac{\partial}{\partial Q^L}N^A_C(Q)|_{Q=Q^*}$.) The variable $\tilde{\mathfrak{X}}$ in equation (11) is defined by the equality: $\sum_{\bar{K}=1}^{n_P}\tilde{\mathfrak{X}}^A_{\bar{K}}(Q^*)\tilde{\mathfrak{X}}^B_{\bar{K}}(Q^*) = G^{AB}(Q^*)$.

The stochastic differential equation for the group variable $a^\alpha(t)$ can be obtained by the same method as was done for the variable $Q^{*A}(t)$. We get the following equation:

$$\begin{aligned}\mathrm{d}a^\alpha = -\frac{1}{2}\mu^2\kappa\bigg[&G^{RS}\tilde{\Gamma}^B_{RS}(Q^*)\Lambda^\beta_B\bar{v}^\alpha_\beta + G^{RP}\Lambda^\sigma_R\Lambda^\beta_B K^B_{\sigma P}\bar{v}^\alpha_\beta - G^{CA}N^M_C\frac{\partial}{\partial Q^{*M}}\big(\Lambda^\beta_A\big)\bar{v}^\alpha_\beta\\ &- G^{MB}\Lambda^\epsilon_M\Lambda^\beta_B\bar{v}^\nu_\epsilon\frac{\partial}{\partial a^\nu}\big(\bar{v}^\alpha_\beta\big)\bigg]\mathrm{d}t + \mu\sqrt{\kappa}\bar{v}^\alpha_\beta\Lambda^\beta_B\tilde{\mathfrak{X}}^B_{\bar{M}}\,\mathrm{d}w^{\bar{M}}\end{aligned} \tag{12}$$

where $\bar{v} \equiv \bar{v}(a)$ and other coefficients depend on Q^*. Also, we have introduced a new notation:

$$\Lambda^\alpha_B = (\Phi^{-1})^\alpha_\mu\chi^\mu_B.$$

In (12), the Christoffel symbols $\tilde{\Gamma}^B_{RS}(Q^*)$ are obtained from $\Gamma^A_{BC}(Q)$, if in its standard definition we transform the derivatives of the metric tensor by formula (5).

In the stochastic differential equation (11) obtained, the drift term has a rather complicated expression. However, it is possible to simplify its expression by using the relation of the drift to the geometrical data.

6. Geometry of the drift coefficient

In the previous section we have obtained a local stochastic differential equation for the process $\zeta^{\mathcal{A}}(t) = (Q^{*A}(t), a^\alpha(t))$. The differential generator of this process is the Laplace–Beltrami operator $\triangle_{\mathcal{P}}$ of (1) expressed in new variables (Q^{*A}, a^α). The drift terms of stochastic differential equation (11) are equal to the terms linear in derivatives of this transformed Laplacian. Hence, we have a correspondence between the stochastic differential equation of the process $\zeta^{\mathcal{A}}(t)$ and its differential generator. Our study of the drift geometry will be based on this correspondence.

For further transformation, it would be very useful for us to regroup the original Laplacian $\triangle_{\mathcal{P}}$ (given in Q variables) in such a way that we will have an opportunity to reveal the geometrical structures of our problem. It was done in [3], where the Laplace–Beltrami operator was presented in the following form:

$$\tfrac{1}{2}\triangle_{\mathcal{P}}(Q) = \tfrac{1}{2}\big({}^\perp G^{AB}\nabla_A\nabla_B - K^A_\mu\gamma^{\mu\nu}\big(\nabla_A K^B_\nu\big)\nabla_B + K^A_\alpha\gamma^{\alpha\beta}\nabla_B K^B_\beta\nabla_B\big) \tag{13}$$

where ${}^\perp G^{AB} = G^{AB} - K^A_\alpha\gamma^{\alpha\beta}K^B_\beta$, and ∇_A is the symbol of the covariant derivative which is obtained with the Christoffel coefficients for the original Riemannian metric $G_{AB}(Q)$.

Now we are interested in the correspondence between the drift terms of the stochastic differential equation (11) and the operator obtained from the Laplace–Beltrami operator (13) after the replacement of the variables Q for the variables (Q^*, a). Since the drift terms of (11) depend on Q^*, we restrict our attention only to the coefficients of the first partial derivatives over Q^* in the transformed Laplacian. It can be shown that the terms (we are interested in) are obtained only from the first and second parts of the Laplacian (13). The sum of these terms is equal to the drift of stochastic differential equation (11).

Therefore, in the drift terms of (11), we select two groups of terms. In accordance with their 'origin', we denote them by $b_{\rm I}^A(Q^*)$ and $b_{\rm II}^A(Q^*)$.

First we consider the geometry of $b_{\rm II}^A(Q^*)$. By performing the necessary calculations, it can be proved that $b_{\rm II}^A(Q^*)$ is given by the projection on the submanifold $\{\chi^\alpha = 0\}$ of the mean curvature vector of the orbit

$$j^D(Q)\frac{\partial}{\partial Q^D} = \frac{1}{2}\Pi^D_A(Q)\gamma^{\alpha\beta}(Q)\big[\nabla_{K_\alpha(Q)}K_\beta(Q)\big]^A\frac{\partial}{\partial Q^D}.$$

The projection operator $\Pi^D_A = \delta^D_A - K^D_\mu\gamma^{\mu\nu}K^C_\nu G_{CA}$ extracts the direction which is normal to the orbit: $\Pi^D_A K^A_\alpha = 0$.

Since we have the Riemannian metric $\tilde{G}_{\mathcal{AB}}$ (together with $\tilde{G}^{\mathcal{AB}}$ which is its inversion), we can form the projection of this mean curvature as follows:

$$\tilde{G}^{SL}\tilde{G}\left(j^D\frac{\partial}{\partial Q^D}, \frac{\partial}{\partial Q^{*S}}\right)\frac{\partial}{\partial Q^{*L}}.$$

Note that before evaluation of the projection we have changed the variables Q in $j^D\frac{\partial}{\partial Q^D}$ for Q^* and a.

As a result, we get the following expression for $b_{\rm II}^A(Q^*)$:

$$b^A_{II}(Q^*) = \tfrac{1}{2}G^{EU}N^A_E N^D_U\big[\gamma^{\alpha\beta}G_{CD}\big(\tilde{\nabla}_{K_\alpha}K_\beta\big)^C\big]$$

in which all the values on the right-hand side depend on Q^* and by $\big(\tilde{\nabla}_{K_\alpha}K_\beta\big)^C(Q^*)$ we denote

$$K^A_\alpha(Q^*)\ \frac{\partial}{\partial Q^A}K^C_\beta(Q)\bigg|_{Q=Q^*} + K^A_\alpha(Q^*)K^B_\beta(Q^*)\tilde{\Gamma}^C_{AB}(Q^*)$$

where

$$\tilde{\Gamma}^C_{AB}(Q^*) = \frac{1}{2}G^{CE}(Q^*)\left(\frac{\partial}{\partial Q^{*A}}G_{EB}(Q^*) + \frac{\partial}{\partial Q^{*B}}G_{EA}(Q^*) - \frac{\partial}{\partial Q^{*E}}G_{AB}(Q^*)\right).$$

Now we proceed to the investigation of the geometry of $b_{\rm I}^A(Q^*)$, which, we recall, is obtained from the first term of the Laplacian expansion in (13). The relation of $b_{\rm I}^A(Q^*)$ to the geometry of the problem can be found as follows.

Note that in the local picture, the projection onto the orbit space $\mathcal{M}$, which is locally isomorphic to Σ, is realized by taking $Q^A = Q^{*A}(x)$ in the expressions that depend on Q. On the orbit space, the first term on the right-hand side of equation (13) (written in (Q^*, a) variables) will be the Laplace–Beltrami operator given on the manifold $(\mathcal{M}, h_{ij})$ with the induced metric

$$h_{ij}(x) = Q^{*A}_{\ \,i}(x)G^H_{AB}(Q^*(x))Q^{*B}_{\ \,j}(x).$$

Here $G^H_{AB} = \Pi^D_A G_{BD}$. (So, the superscript H will mean horizontality.)

In operators, the transition from Σ to $\mathcal{M}$ is obtained by using the replacement of the derivatives: $\frac{\partial}{\partial Q^{*A}} = T^i_A\frac{\partial}{\partial x^i}$, where $T^i_A = (P_\perp)^S_A G^H_{SE}Q^{*E}_{\ \,k}(x)h^{ki}$ (here $P_\perp$ and G^H depend on $Q^*(x)$). An inverse replacement is given by $\frac{\partial}{\partial x^i} = Q^{*A}_{\ \,i}(x)\frac{\partial}{\partial Q^{*A}}$.

The orbit space diffusion is locally described by the following stochastic differential equation:

$$\mathrm{d}x^i(t) = -\tfrac{1}{2}\mu^2\kappa h^{kl}(x(t))\Gamma^i_{kl}(x(t))\,\mathrm{d}t + \mu\sqrt{\kappa}X^i_{\bar{m}}(x(t))\,\mathrm{d}w^{\bar{m}}(t)$$

in which the Christoffel coefficients correspond to the metric $h_{ij}(x)$.

By all these transformations we come to the following chain of correspondences. The drift of the above stochastic differential equation is related to the coefficients at the first partial derivative over x in the Laplace–Beltrami operator on $\mathcal{M}$. On Σ, this drift is presented by the corresponding coefficients at the first partial derivative over Q^* in the expression obtained from the first term of the Laplacian (13). And also, we have already established that this coefficient is equal to b_{I}.

Therefore, we have reduced our problem of the geometrical representation for b_{I} to the same problem but for the drift of the stochastic differential equation given on $\mathcal{M}$.

It is known that if a submanifold is embedded into some external manifold, we can describe diffusion on this submanifold not only by means of the internal variables (given on a submanifold) but also with the external variables. A particular case of such a description was considered in [17], where the Euclidean space was taken as an external manifold. It is not difficult to find a similar description for a general case (see the appendix).

Note that the orbit space is a submanifold in the (Riemannian) manifold $\big(\mathcal{P}, G^H_{AB}(Q)\big)$ with the degenerate metric G^H_{AB}. Therefore, in our case, we shall also use the method from the appendix for the derivation of the stochastic differential equations that are given in the external variables.

But we should slightly correct the calculations from the appendix. In formulae from the appendix, we must change the metric G_{AB} for the degenerate metric G^H_{AB}. As a consequence, instead of relation (A.4) in the appendix we will have

$$h^{kl}(x)\Gamma^i_{kl} = G^H_{AB}\big(Q^{*A}_{kl}h^{kl} + {}^H\Gamma^A_{CD}Q^{*C}_{k}Q^{*D}_{l}h^{kl}\big)h^{im}Q^{*B}_{m}$$

where $G^H_{AB}(Q^*(x))^H\Gamma^B_{CD}(Q^*(x))$ is defined as

$$G^H_{AB}\,{}^H\Gamma^B_{CD} = \tfrac{1}{2}\big(G^H_{AC,D} + G^H_{AD,C} - G^H_{CD,A}\big). \tag{14}$$

In (14), by derivatives we mean the following: $G^H_{AC,D} \equiv \frac{\partial G^H_{AC}(Q)}{\partial Q^D}\big|_{Q=Q^*(x)}$.

As a result of our calculation, we get the following expression for drift of the orbit space stochastic differential equation:

$$b^A_{\mathrm{I}}(Q^*(x)) = -\tfrac{1}{2}G^{EM}(Q^*(x))N^C_E(Q^*(x))N^B_M(Q^*(x))^H\Gamma^A_{CB}(Q^*(x)) + j^A_{\mathrm{I}}$$

where j_{I} is the mean curvature vector of the orbit space. It can be evaluated as follows:

$$j^A_{\mathrm{I}} = \tfrac{1}{2}\left(\delta^A_B - N^A_B(Q^*(x))\right)h^{ij}(x)\big[Q^{*B}_{ij} + Q^{*P}_{i}Q^{*L}_{j}\,{}^H\Gamma^B_{PL}(Q^*(x))\big].$$

It was established earlier, that this drift coincides with b_{I}. It is for this reason that we use this notation for the drift of the stochastic differential equation given on the orbit space.

As a function, the mean curvature of the orbit space is given on a submanifold. In the same way as done in the appendix, we can redefine the stochastic variables $Q^*(x(t))$ for new stochastic variables $Q^*(t)$. (We denote new stochastic variables by the same letter.)

Note that in equation (14), the Christoffel symbols ${}^H\Gamma^B_{CD}$ are defined up to the terms $\tilde{T}^B_{CD}$ that are satisfied to $G^H_{AB}\tilde{T}^B_{CD} = 0$. However, this ambiguity is not essential for us, since b_{I} can also be presented as

$$b^A_{\mathrm{I}} = -\tfrac{1}{2}N^A_E\,{}^H\Gamma^E_{CD}N^C_K N^D_U G^{KU} + \tfrac{1}{2}N^A_{LM}N^L_K N^M_U G^{KU}.$$

Thus, now we know the relation of b_I and b_{II} to the geometrical values. And it permits us to rewrite equation (8) as follows:

$$\mathrm{d}Q^{*A}(t) = \mu^2\kappa\big(-\tfrac{1}{2}G^{EM}N^C_E N^B_M\,{}^H\Gamma^A_{CB} + j^A_{\mathrm{I}} + j^A_{\mathrm{II}}\big)\,\mathrm{d}t + \mu\sqrt{\kappa}N^A_C\tilde{\mathfrak{X}}^C_{\bar{M}}\,\mathrm{d}w^{\bar{M}} \tag{15}$$

where all the values on the right-hand side depend on $Q^*(t)$. In the above equation, we have introduced a new notation for b^A_{II}. It was denoted by j^A_{II}.

The solutions of the stochastic differential equations (15) and (12) are local representatives of the stochastic process $\zeta(t)$. In charts of the manifold, a set of the solutions of these equations determines the local stochastic evolution families of mappings of the manifold $\mathcal{P}$. As in [11], with these local families it is possible to define the global stochastic process $\zeta(t)$ which consists of two components related, correspondingly, to the stochastic evolution on the submanifold Σ (the gauge surface) and to the stochastic evolution on the orbit of the principal fibre bundle.

The transformation of the stochastic process $\eta(t)$, considered in the previous sections, leads to the corresponding transformation of the global semigroup (4). Now, the global semigroup is determined by the superposition of the local semigroups $\tilde{U}_{\zeta^{\varphi^P}}$:

$$\psi_{t_b}(p_a, t_a) = \lim{}_q \tilde{U}_{\zeta^{\varphi^P}}(t_a, t_1) \cdots \tilde{U}_{\zeta^{\varphi^P}}(t_{n-1}, t_b)\tilde{\phi}_0(Q^*_a, \theta_a) \tag{16}$$

where

$$\tilde{U}_{\zeta^{\varphi^P}}(s,t)\tilde{\phi}_0(Q^*_0, \theta_0) = \mathrm{E}_{s,(Q^*_0,\theta_0)}\tilde{\phi}_0(Q^*(t), a(t)) \qquad Q^*(s) = Q^*_0 \quad a(s) = \theta_0.$$

We will write this global semigroup in the following symbolic form:

$$\psi_{t_b}(p_a, t_a) = \mathrm{E}\left[\tilde{\phi}_0(\xi_\Sigma(t_b), a(t_b)) \exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b} \tilde{V}(\xi_\Sigma(u))\,\mathrm{d}u\right\}\right]$$

where $\xi_\Sigma(t_a) = Q^*_a$, $a(t_a) = \theta_a$, $\varphi^P(p_a) = (Q^*_a, \theta_a)$. In this formula we have taken into account an earlier omitted potential term.

It follows from (15) and (12) that the coordinate representation of the differential generator of the semigroup related to the stochastic process $\zeta(t)$ is given by

$$\begin{aligned}
\frac{1}{2}\mu^2\kappa \Bigg(& G^{CD}N^A_C N^B_D \frac{\partial^2}{\partial Q^{*A}\partial Q^{*B}} - G^{CD}N^A_C N^{BH}_D \Gamma^E_{AB}\frac{\partial}{\partial Q^{*E}} + j^A_{\mathrm{I}}\frac{\partial}{\partial Q^{*A}} \\
& + j^A_{\mathrm{II}}\frac{\partial}{\partial Q^{*A}} + G^{AB}\Lambda^\alpha_A\Lambda^\beta_B \bar{L}_\alpha \bar{L}_\beta - G^{RS}\tilde{\Gamma}^B_{RS}\Lambda^\alpha_B\bar{L}_\alpha - G^{RP}\Lambda^\sigma_R\Lambda^\alpha_B K^B_{\sigma P}\bar{L}_\alpha \\
& + G^{CA}N^M_C\frac{\partial}{\partial Q^{*M}}\left(\Lambda^\alpha_A\right)\bar{L}_\alpha + 2G^{BC}N^A_C\Lambda^\alpha_B\bar{L}_\alpha\frac{\partial}{\partial Q^{*A}}\Bigg) + \frac{1}{\mu^2\kappa m}\tilde{V}.
\end{aligned}$$

Here all the values, except for $\bar{L}$, depend on Q^*.

7. Factorization of the path integral measure

In [4, 5], a new method of factorization of the path integral measure was proposed. The main idea of [4] was to exploit the stochastic differential equation in the nonlinear filtering theory [18, 19]. This equation describes the evolution of the conditional mathematical expectation of the signal process (the process $a(t)$ in our case) with respect to the σ algebra generated by an observable process (the stochastic process $Q^*(t)$).

In the global semigroup (16) obtained, each local semigroup $\tilde{U}_{\zeta^{\varphi^P}}$ can be presented as follows:

$$\tilde{U}_{\zeta^{\varphi^P}}(s,t)\tilde{\phi}(Q^*_0, \theta_0) = \mathrm{E}\left[\mathrm{E}\left[\tilde{\phi}(Q^*(t), a(t)) \mid (\mathcal{F}_{Q^*})^t_s\right]\right]. \tag{17}$$

This transformation is based on the properties of the conditional expectation of the Markov processes. The above path integral transformation can also be viewed as an analogue of the transition from the multiple integrals to the repeated ones in the ordinary integration.

The conditional mathematical expectation on the right-hand side of (17),

$$\hat{\tilde{\phi}}(Q^*(t)) \equiv \mathrm{E}\left[\tilde{\phi}(Q^*(t), a(t)) \mid (\mathcal{F}_{Q^*})^t_s\right]$$

should satisfy the nonlinear filtering equation. We use the definition of this equation in [18, 19]. In our case, it is derived by using the stochastic differential equations (15) and (12). The nonlinear filtering stochastic differential equation of our problem is as follows:

$$\begin{aligned}\mathrm{d}\hat{\tilde{\phi}} = &-\frac{1}{2}\mu^2\kappa\left(G^{RS}\tilde{\Gamma}^B_{RS}\Lambda^\beta_B + G^{RP}\Lambda^\sigma_R\Lambda^\beta_B K^B_{P\sigma} - G^{CA}N^M_C\frac{\partial}{\partial Q^{*M}}\left(\Lambda^\beta_A\right)\right)\\ &\times \mathrm{E}\left[\bar{L}_\beta\tilde{\phi}\mid(\mathcal{F}_{Q^*})^t_s\right]\mathrm{d}t + \frac{1}{2}\mu^2\kappa G^{CB}\Lambda^\nu_C\Lambda^\kappa_B\mathrm{E}\left[\bar{L}_\nu\bar{L}_\kappa\tilde{\phi}\mid(\mathcal{F}_{Q^*})^t_s\right]\mathrm{d}t\\ &+\mu\sqrt{\kappa}\Lambda^\beta_C\Pi^C_K\tilde{\mathfrak{X}}^K_{\bar{M}}\mathrm{E}\left[\bar{L}_\beta\tilde{\phi}\mid(\mathcal{F}_{Q^*})^t_s\right]\mathrm{d}w^{\bar{M}}.\end{aligned}\tag{18}$$

It is possible to perform a separation of the variables in this equation. It can be done by applying the Peter–Weyl theorem to the function $\widetilde{\phi}$ considered as a function given on a group $\mathcal{G}$. Using this theorem, the function $\widetilde{\phi}$ can be presented as

$$\tilde{\phi}(Q^*, a) = \sum_{\lambda,p,q} c^\lambda_{pq}(Q^*)D^\lambda_{pq}(a)$$

where $D^\lambda_{pq}(a)$ are the matrix elements of an irreducible representation T^λ of a group $\mathcal{G}$: $\sum_q D^\lambda_{pq}(a)D^\lambda_{qn}(b) = D^\lambda_{pn}(ab)$.

Using the properties of the conditional mathematical expectations, we have

$$\begin{aligned}\mathrm{E}\left[\tilde{\phi}(Q^*(t), a(t))\mid(\mathcal{F}_{Q^*})^t_s\right] &= \sum_{\lambda,p,q} c^\lambda_{pq}(Q^*(t))\mathrm{E}\big[D^\lambda_{pq}(a(t))\mid(\mathcal{F}_{Q^*})^t_s\big]\\ &\equiv \sum_{\lambda,p,q} c^\lambda_{pq}(Q^*(t))\hat{D}^\lambda_{pq}(Q^*(t))\end{aligned}$$

where

$$c^\lambda_{pq}(Q^*(t)) = d^\lambda\int_{\mathcal{G}}\tilde{\varphi}(Q^*(t),\theta)\bar{D}^\lambda_{pq}(\theta)\,\mathrm{d}\mu(\theta)$$

d^λ is the dimension of an irreducible representation and $\mathrm{d}\mu(\theta)$ is a normalized $\left(\int_{\mathcal{G}}\mathrm{d}\mu(\theta) = 1\right)$ invariant Haar measure on a group $\mathcal{G}$.

Then, as in [4, 5], we get the stochastic differential equation for the conditional mathematical expectation $\hat{D}^\lambda_{pq}$:

$$\begin{aligned}\mathrm{d}\hat{D}^\lambda_{pq}(Q^*(t)) = &-\frac{1}{2}\mu^2\kappa\left\{\left[G^{RS}\tilde{\Gamma}^B_{RS}\Lambda^\mu_B + G^{RP}\Lambda^\sigma_R\Lambda^\mu_B K^B_{P\sigma}\right.\right.\\ &\left.-G^{CA}N^M_C\frac{\partial}{\partial Q^{*M}}\left(\Lambda^\beta_A\right)\right](J_\beta)^\lambda_{pq'}\hat{D}^\lambda_{q'q}(Q^*(t))\\ &\left.-G^{CB}\Lambda^\alpha_C\Lambda^\nu_B(J_\alpha)^\lambda_{pq'}(J_\nu)^\lambda_{q'q''}\hat{D}^\lambda_{q''q}(Q^*(t))\right\}\mathrm{d}t\\ &+\mu\sqrt{\kappa}\Lambda^\nu_C\Pi^C_K(J_\nu)^\lambda_{pq'}\hat{D}^\lambda_{q'q}(Q^*(t))\tilde{\mathfrak{X}}^K_{\bar{M}}(Q^*(t))\,\mathrm{d}^{\bar{M}}(t)\end{aligned}\tag{19}$$

in which $(J_\mu)^\lambda_{pq} \equiv \left(\frac{\partial D^\lambda_{pq}(a)}{\partial a^\mu}\right)\Big|_{a=e}$ are the infinitesimal generators of the representation $D^\lambda(a)$:

$$\bar{L}_\mu D^\lambda_{pq}(a) = \sum_{q'}(J_\mu)^\lambda_{pq'}D^\lambda_{q'q}(a).$$

We note that the conditional expectation $\hat{D}^\lambda_{pq}(Q^*(t))$ also depends on the initial points $Q^*_0 = Q^*(s)$ and $\theta^\alpha_0 = a^\alpha(s)$.

The solution of the linear matrix stochastic differential equation (19) can be written [20] as follows:

$$\hat{D}^\lambda_{pq}(Q^*(t)) = (\overleftarrow{\exp})^\lambda_{pn}(Q^*(t), t, s)\mathrm{E}\left[D^\lambda_{nq}(a(s))\mid(\mathcal{F}_{Q^*})^t_s\right]\tag{20}$$

where

$$(\overleftarrow{\exp})^{\lambda}_{pn}(Q^*(t),t,s)=\overleftarrow{\exp}\int_s^t\left\{\frac{1}{2}\mu^2\kappa\left[\bar{\gamma}^{\sigma\nu}(Q^*(u))(J_\sigma)^\lambda_{pr}(J_\nu)^\lambda_{rn}\right.\right.$$
$$\left.-\left(G^{RS}\tilde{\Gamma}^B_{RS}\Lambda^\beta_B+G^{RP}\Lambda^\sigma_R\Lambda^\beta_B K^B_{P\sigma}-G^{CA}N^M_C\frac{\partial}{\partial Q^{*M}}\left(\Lambda^\beta_A\right)\right)(J_\beta)^\lambda_{pn}\right]\mathrm{d}u$$
$$\left.+\mu\sqrt{\kappa}\Lambda^\beta_C(J_\beta)^\lambda_{pn}\Pi^C_K\tilde{\mathfrak{X}}^K_{\bar{M}}\,\mathrm{d}w^{\bar{M}}\right\} \tag{21}$$

is a multiplicative stochastic integral. This integral is a limit of the sequence of time-ordered multipliers that have been obtained as a result of breaking a time interval $[s,t]$, $[s=t_0\leqslant t_1\cdots\leqslant t_n=t]$. In (21), the time order of these multipliers is indicated by the arrow directed to the multipliers given at greater times.

Taking into account the representation for $\hat{D}^\lambda_{pq}$ obtained in (20) and (21), we rewrite our local semigroup (17) as follows:

$$\tilde{U}_{\zeta^{\varphi^P}}(s,t)\tilde{\phi}(Q^*_0,\theta_0)=\sum_{\lambda,p,q,q'}\mathrm{E}\left[c^\lambda_{pq}(Q^*(t))(\overleftarrow{\exp})^\lambda_{pq'}(Q^*(t),t,s)\right]D^\lambda_{q'q}(\theta_0) \tag{22}$$

where we have taken into account that

$$\mathrm{E}\left[D^\lambda_{nq}(a(s))\mid(\mathcal{F}_{Q^*})^t_s\right]=D^\lambda_{nq}(a(s))=D^\lambda_{nq}(\theta_0).$$

In order to obtain the global semigroup by the methods in [11], one should take the piecewise breaking of the time interval $[t_a,t_b]$, $[t_a=t_0\leqslant t_1\cdots\leqslant t_n=t_b]$ and then take the superposition of the local semigroups that are similar to (22). The global semigroup for the global process is obtained as a limit (under the refinement of the subdivision of the time interval) of the superposition of these local semigroups.

We write the result of the limiting procedure, which is the relation between the global semigroups, in the following symbolic form:

$$\psi_{t_b}(p_a,t_a)=\sum_{\lambda,p,q,q'}\mathrm{E}\left[c^\lambda_{pq}(\xi_\Sigma(t_b))(\overleftarrow{\exp})^\lambda_{pq'}(\xi_\Sigma(t),t_b,t_a)\right]D^\lambda_{q'q}(\theta_a)\qquad(\xi_\Sigma(t_a)=\pi|_\Sigma\circ p_a) \tag{23}$$

where $\xi_\Sigma(t)$ is a global stochastic process defined on the submanifold Σ. This process is described locally by equations (15).

Thus, our initial original path integral has been rewritten as the sum of the matrix semigroups (the path integrals) that are given on the submanifold Σ. The differential generator (the Hamiltonian operator) of these matrix semigroups is

$$\frac{1}{2}\mu^2\kappa\left\{\left[G^{CD}N^A_C N^B_D\frac{\partial^2}{\partial Q^{*A}\partial Q^{*B}}-G^{CD}N^E_C N^{MH}_D\Gamma^A_{EM}\frac{\partial}{\partial Q^{*A}}+\left(j^A_{\mathrm{I}}+j^A_{\mathrm{II}}\right)\frac{\partial}{\partial Q^{*A}}\right](I^\lambda)_{pq}\right.$$
$$+2N^A_C G^{CP}\Lambda^\alpha_P(J_\alpha)^\lambda_{pq}\frac{\partial}{\partial Q^{*A}}-\left(G^{RS}\tilde{\Gamma}^B_{RS}\Lambda^\alpha_B+G^{RP}\Lambda^\sigma_R\Lambda^\alpha_B K^B_{P\sigma}\right.$$
$$\left.\left.-G^{CA}N^M_C\frac{\partial}{\partial Q^{*M}}\left(\Lambda^\alpha_A\right)\right)(J_\alpha)^\lambda_{pq}+G^{SB}\Lambda^\alpha_B\Lambda^\sigma_S(J_\alpha)^\lambda_{pq'}(J_\sigma)^\lambda_{q'q}\right\} \tag{24}$$

where $(I^\lambda)_{pq}$ is a unity matrix.

The operator acts in the space of the section $\Gamma(\Sigma,V^*)$ of the associated covector bundle. The scalar product in the space of the sections $\Gamma(\Sigma,V^*)$ is defined as follows:

$$(\psi_n,\psi_m)=\int_\Sigma\langle\psi_n,\psi_m\rangle_{V^*_\lambda}\frac{\det\Phi^\alpha_\beta}{\det^{1/2}\left(\chi^\mu_A G^{AB}\chi^\nu_B\right)}\,\mathrm{d}v_\Sigma \tag{25}$$

where $\mathrm{d}v_\Sigma$ is the Riemannian volume element on Σ.

An integration measure of the scalar product in formula (25) has been obtained from the Riemannian volume element of the manifold $\mathcal{P}$, in which we have changed the variables Q^A for (Q^{*i}, a^α). Also, we make use of the equality:

$$\det G_{AB}(Q^{*i}, Q^{*\alpha}(Q^{*i}), a^\mu) = \det((G_\Sigma)_{AB}) \det{}^{-1}\big(\big(G^{BC}\chi^\nu_B\chi^\mu_C\big)\Phi^{-1}{}^\alpha_\mu\Phi^{-1}{}^\beta_\nu\bar{v}^\sigma_\alpha\bar{v}^\rho_\beta\big).$$

Note that the metric $(G_\Sigma)_{AB}(Q^{*i}, Q^{*\alpha}(Q^{*i}))$ is a restriction of the metric $(P_\perp)^C_A(Q^*)G_{CD}(Q^*)(P_\perp)^D_B(Q^*)$ to the surface Σ.

Performing the transformation of the measure in integral (25), we can also present the scalar product in the following form:

$$(\psi_n, \psi_m) = \int \langle\psi_n, \psi_m\rangle_{V^*_\lambda} \det \Phi^\alpha_\beta \prod_{\alpha=1}^{N_\mathcal{G}} \delta(\chi^\alpha(Q^*)) \det{}^{1/2} G_{AB}\, \mathrm{d}Q^{*1} \wedge \cdots \wedge \mathrm{d}Q^{*N_\mathcal{P}}.$$

In order to get the representation of the semigroups (the path integrals) that are given under the sign of the sum in (23) in terms of the semigroups on $\mathcal{P}$, we should take the inverse of equality (23). As in [5], we will do it for the kernels of the corresponding local semigroups.

Provided that the necessary restrictions are fulfilled, the semigroup on the left-hand side of (23) has the kernel and can be presented as

$$\psi_{t_b}(p_a, t_a) = \int G_\mathcal{P}(p_b, t_b; p_a, t_a)\phi_0(p_b)\, \mathrm{d}v_\mathcal{P}(p_b). \tag{26}$$

Using the partition of the unity subordinated to a local finite covering of the manifold $\mathcal{P}$ and keeping in mind that there is a local isomorphism of $P_\Sigma(\Sigma, \mathcal{G})$ with the trivial principal fibre bundle $\varphi^\Sigma_{\alpha_b}\big(\mathcal{U}^\Sigma_{\alpha_b}\big) \times \mathcal{G}$, by which a chart in the atlas of the manifold $\mathcal{P}$ is changed for the chart $\varphi^\Sigma_{\alpha_b}\big(\mathcal{U}^\Sigma_{\alpha_b}\big) \times \mathcal{G}$, we obtain the following expression for the right-hand side of (26):

$$\sum_{\alpha_b} \int_{\varphi^\Sigma_{\alpha_b}(\mathcal{U}^\Sigma_{\alpha_b})\times\mathcal{G}} \tilde{\bar{\mu}}_{\alpha_b}(x_b) G_\mathcal{P}(\alpha_b, F(Q^*_b, \theta_b), t_b; \beta_a, F(Q^*_a, \theta_a), t_a)\tilde{\phi}_0(Q^*_b, \theta_b)\, \mathrm{d}v(Q^*_b)\, \mathrm{d}\mu(\theta_b) \tag{27}$$

where $\mathrm{d}v(Q^*)$ is the same volume measure as in (25) and $\mathrm{d}\mu(\theta) = \det \bar{u}^\alpha_\beta(\theta)\, \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^{N_\mathcal{G}}$ is a Haar measure on a group $\mathcal{G}$.

A local representation of the right-hand side of (23) can also be given as

$$\sum_{\alpha_b} \int_{\varphi^\Sigma_{\alpha_b}(\mathcal{U}^\Sigma_{\alpha_b})} \tilde{\rho}_{\alpha_b}(x_b) \sum_{\lambda,p,q,q'} G^\lambda_{q'p}(\alpha_b, Q^*_b, t_b; \beta_a, Q^*_a, t_a)c^\lambda_{pq}(Q^*_b)D^\lambda_{q'q}(\theta_a)\, \mathrm{d}v(Q^*_b). \tag{28}$$

Comparing (27) and (28), we find the relation between the local Green functions:

$$\begin{aligned}\int_\mathcal{G} & G_\mathcal{P}(\alpha_b, F(Q^*_b, \theta_b), t_b; \beta_a, F(Q^*_a, \theta_a), t_a)D^\lambda_{pq}(\theta_b)\, \mathrm{d}\mu(\theta_b) \\ &= \sum_{q'} G^\lambda_{q'p}(\alpha_b, Q^*_b, t_b; \beta_a, Q^*_a, t_a)D^\lambda_{q'q}(\theta_a)\end{aligned}$$

which, by the unimodularity of the group $\mathcal{G}$, can be easily reversed:

$$G^\lambda_{mn}(\alpha_b, Q^*_b, t_b; \beta_a, Q^*_a, t_a) = \int_\mathcal{G} G_P(\alpha_b, Q^*_b, \theta, t_b; \beta_a, Q^*_a, e, t_a)D^\lambda_{nm}(\theta)\, \mathrm{d}\mu(\theta). \tag{29}$$

In this formula, e corresponds to the unity element of a group $\mathcal{G}$ and

$$G_P(\alpha_b, Q^*_b, \theta_b, t_b; \beta_a, Q^*_a, \theta_a, t_a) \equiv G_\mathcal{P}(\alpha_b, F(Q^*_b, \theta_b), t_b; \beta_a, F(Q^*_a, \theta_a), t_a).$$

In the case of the trivial principal fibre bundle, gluing of these local Green functions can be done with the help of the transition coordinate functions defined in the manifold atlas. It gives rise to the global Green function.

Therefore, equality (29) can be extended from the local charts to the whole manifold and we obtain the relation between the Green function defined on the global manifolds:

$$G^{\lambda}_{mn}(\pi_\Sigma(p_b), t_b; \pi(p_a), t_a) = \int_{\mathcal{G}} G_{\mathcal{P}}(p_b\theta, t_b; p_a, t_a) D^{\lambda}_{nm}(\theta)\, \mathrm{d}\mu(\theta). \tag{30}$$

The path integral on the left-hand side of this equality can be written symbolically as

$$\begin{aligned}
&G^{\lambda}_{mn}(\pi_\Sigma(p_b), t_b; \pi_\Sigma(p_a), t_a) \\
&\quad= \tilde{\mathrm{E}}_{\substack{\xi_\Sigma(t_a)=\pi_\Sigma(p_a)\\ \xi_\Sigma(t_b)=\pi_\Sigma(p_b)}}\left[(\overleftarrow{\exp})^{\lambda}_{mn}(\xi_\Sigma(t), t_b, t_a)\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(\xi_\Sigma(u))\,\mathrm{d}u\right\}\right] \\
&\quad= \int\limits_{\substack{\xi_\Sigma(t_a)=\pi_\Sigma(p_a)\\ \xi_\Sigma(t_b)=\pi_\Sigma(p_b)}} \mathrm{d}\mu^{\xi_\Sigma}\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(\xi_\Sigma(u))\,\mathrm{d}u\right\} \\
&\qquad\times\overleftarrow{\exp}\int_{t_a}^{t_b}\left\{\frac{1}{2}\mu^2\kappa\left[\gamma^{\sigma\nu}(\xi_\Sigma(u))(J_\sigma)^{\lambda}_{mr}(J_\nu)^{\lambda}_{rn}\right.\right. \\
&\qquad-\left.\left(G^{RS}\tilde{\Gamma}^{B}_{RS}\Lambda^{\beta}_{B} + G^{RP}\Lambda^{\sigma}_{R}\Lambda^{\beta}_{B}K^{B}_{P\sigma} - G^{CA}N^{M}_{C}\frac{\partial}{\partial Q^{*M}}\left(\Lambda^{\beta}_{A}\right)\right)(J_\beta)^{\lambda}_{mn}\right]\mathrm{d}u \\
&\qquad+\left.\mu\sqrt{\kappa}\Lambda^{\beta}_{C}(J_\beta)^{\lambda}_{mn}\Pi^{C}_{K}\tilde{\mathfrak{X}}^{K}_{\bar{M}}\,\mathrm{d}w^{\bar{M}}\right\}.
\end{aligned} \tag{31}$$

The semigroup, given by this kernel, acts in the space of the equivariant functions:

$$\tilde{\psi}_n(pg) = D^{\lambda}_{mn}(g)\tilde{\psi}_m(p).$$

$\tilde{\psi}_n$ are isomorphic to the functions ψ_n that belong to the space of the sections $\Gamma(\Sigma, V^*)$ of the associated covector bundle:

$$\tilde{\psi}_n(F(Q^*, e)) = \psi_n(Q^*).$$

The method, by which we have obtained the integral relation between G^{λ}_{mn} and $G_{\mathcal{P}}$, can be considered as the realization of the reduction procedure in the path integrals for the dynamical systems with a symmetry.

The reduction to the zero-momentum level, i.e., when $\lambda = 0$, establishes the relation between the path integrals that are used for descriptions of the quantum motion of the scalar particle on an initial manifold $\mathcal{P}$ and on the the orbit space manifold $\mathcal{M}$.

In our case, in order to represent the motion on the orbit space, we have used an additional gauge surface Σ, on which the corresponding diffusion was given by the stochastic differential equation (15). In this equation, in the drift, there is an 'extra' term j_{II}, which has no direct relation to the orbit space $\mathcal{M}$.

Note that this term completely corresponds to the 'extra' term in the stochastic differential equation from [4, 5]. It can be shown that this term is related to the volume of the orbit.

Without the j_{II} term in the drift, we would have the stochastic process which describes a 'pure' diffusion. After changing the dependent coordinates Q^* for the independent coordinates x^i, the differential generator of a diffusion becomes the Laplace–Beltrami operator given on the orbit space $\mathcal{M}$.

In the path integrals, the transformation in which we change the stochastic process ξ_Σ with the local stochastic differential equation (15) for the process $\tilde{\xi}_\Sigma$, with the stochastic differential equation

$$\mathrm{d}Q^{*A}(t) = \mu^2\kappa\left(-\tfrac{1}{2}G^{EM}N^{C}_{E}N^{B}_{M}{}^{H}\Gamma^{A}_{CB} + j^{A}_{\mathrm{I}}\right)\mathrm{d}t + \mu\sqrt{\kappa}N^{A}_{C}\tilde{\mathfrak{X}}^{C}_{\bar{M}}\,\mathrm{d}w^{\bar{M}} \tag{32}$$

can be made with the help of the Girsanov transformation formula.

The presence of the projection operators in diffusion matrices of equations (15) and (32) tells us that we have the degenerate diffusion matrices. It does not permit us to apply the standard Girsanov formula.

But we can still derive the Girsanov formula if we confine ourselves by the scope of the predefined ambiguities, which originate from the use of the projection operators. As in the ordinary case, the derivation will also be based on the uniqueness (modulo the above ambiguity) of the solution of the parabolic differential equation with the operator given by the diagonal part of the operator (24). One can obtain the Girsanov formula by using the Itô differentiation formula for the composite function and the following equality:

$$\left(G^{AB}N_A^C N_B^D\right)\left((P_\perp)_D^E G_{EM}^H (P_\perp)_L^M\right) = (P_\perp)_L^C.$$

The Radon–Nicodim derivative of the measure μ^{ξ_Σ} with respect to the measure $\mu^{\tilde{\xi}_\Sigma}$ will be as follows:

$$\frac{\mathrm{d}\mu^{\xi_\Sigma}}{\mathrm{d}\mu^{\tilde{\xi}_\Sigma}}(\tilde{\xi}_\Sigma(t)) = \exp\int_{t_a}^{t}\left[-\frac{1}{2}\mu^2\kappa\left((P_\perp)_A^L G_{LK}^H (P_\perp)_E^K\right) j_{II}^A j_{II}^E\,\mathrm{d}t + \mu\sqrt{\kappa}G_{LK}^H (P_\perp)_A^L j_{II}^A \tilde{\mathfrak{X}}_{\bar{M}}^K\,\mathrm{d}w^{\bar{M}}\right].$$

Performing such a change of the integration variables in that path integral which is obtained as a result of the reduction to $\lambda = 0$ momentum level, we get the following integral relation:

$$G_\Sigma(Q_b^*, t_b; Q_a^*, t_a) = \int_{\mathcal{G}} G_{\mathcal{P}}(p_b\theta, t_b; p_a, t_a)\,\mathrm{d}\mu(\theta)$$

where the kernel G_Σ is presented by the path integral

$$\begin{aligned}G_\Sigma(Q_b^*, t_b; Q_a^*, t_a) = &\int_{\substack{\tilde{\xi}_\Sigma(t_a)=Q_a^*\\ \tilde{\xi}_\Sigma(t_b)=Q_b^*}} \mathrm{d}\mu^{\tilde{\xi}_\Sigma}\exp\left\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b} V(\tilde{\xi}_\Sigma(u))\,\mathrm{d}u\right\}\\ &\times\exp\int_{t_a}^{t_b}\left\{-\frac{1}{8}\mu^2\kappa G^{AB}N_A^D N_B^L\left[\gamma^{\alpha\beta}G_{DC}\left(\tilde{\nabla}_{K_\alpha}K_\beta\right)^C\right]\left[\gamma^{\mu\nu}G_{LE}\left(\tilde{\nabla}_{K_\mu}K_\nu\right)^E\right]\mathrm{d}t\right.\\ &\left.+\frac{1}{2}\mu\sqrt{\kappa}N_P^D\left[\gamma^{\alpha\beta}G_{CD}\left(\tilde{\nabla}_{K_\alpha}K_\beta\right)^C\right]\tilde{\mathfrak{X}}_{\bar{M}}^P\,\mathrm{d}w^{\bar{M}}\right\}\qquad (Q^* = \pi_\Sigma(p)).\end{aligned}$$

The semigroup, determined by this path integral, acts in the space of the scalar functions given on Σ.

In the obtained formula, the reduction Jacobian has an additional stochastic integral. It is possible to get rid of this stochastic integral with the help of the corresponding Itô identity. But, as it requires further investigation, we do not give this transformation in the present paper.

Finally we note that in spite of the existing difference in form between the ultimate formula obtained here and an analogous formula from [5], these formulae are equivalent to each other. To verify this fact, one should compare the differential generators of their stochastic processes. Changing the dependent coordinates Q^* for independent one x^i in the differential generator for ξ_Σ gives the differential generator in [5] for the process given on the orbit space.

8. Conclusion

From the path integral transformation considered in the paper, it follows that the path integral measure is not invariant under reduction (formulae (30) and (31)). Earlier, in [5], we studied the path integral reduction in the case when the orbit space manifold was described in terms of independent coordinates. These two approaches give, in fact, the same result expressed in different forms. By the replacement of the coordinates, the corresponding path integrals can be transformed to each other.

The obtained reduction Jacobian reveals an interesting geometrical structure. Namely, it is related to the mean curvature vector of the orbit over the point belonging to the base space in the principal fibre bundle. We obtained this mean curvature term and the term related to the mean curvature of the orbit space by considering the transformation of the stochastic differential equation.

The roles of these mean curvature terms are different. The first mean curvature leads to the Jacobian. The mean curvature of the orbit together with the Christoffel coefficient term makes up the the standard drift term of the stochastic differential equation given on a submanifold. It needs further investigation to elucidate the question of the origin of these mean curvature terms. It may be supposed that they come from the mean curvature of the manifold $\mathcal{P}$ if we view this manifold as the submanifold in some manifold with a larger dimension.

The noninvariance of the path integral measure under reduction was obtained in many papers. Usually, the authors define the path integral with the help of some discrete approximation for the corresponding kernels. Then, the appearance of the reduction Jacobian is explained by the choice of the 'true' operator ordering together with the use of the Faddeev–Popov trick. But this trick does not retain the group integration. Unlike the work [3], our formulae have this integration. But the subject of consideration of [3] was the path integral for the partition function. The standard approach to the representation of the Yang–Mills partition function also does not have the group integration.

On the other hand, the necessity of the group integration in quantum mechanical path integral reduction is verified by the path integral reduction performed for the homogeneous space G/H [2] and by evaluations in concrete examples.

It is worth noting that in the functional integration of the quantum field theory there is an approach [23] which is similar (regarding the presence of the group integration) to the quantum mechanical path integral reduction.

Comparing our Jacobian with the Jacobian of the first authors in [2], we see that their Jacobian (the Hamilton operator) is a particular case of ours. Our Hamiltonian has more general dependence in part related to the Casimir operator. In fact, this comparision should be done not with our present paper, but with [4, 5], since the authors deal with the independent coordinate description of the orbit space. The Jacobian of our previous paper (in independent variables) coincides with that in [3].

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Appendix. Stochastic differential equation on a submanifold

Let the manifold $\mathcal{M}$ be embedded into some smooth (compact) finite dimensional Riemannian manifold with a metric $G_{AB}(Q)$. We assume that this embedding is locally given by the equations $Q^A = Q^A(x^i)$, where $\{Q^A\}$ is a coordinate system on the external manifold and the coordinate system $\{x^i\}$ is given on $\mathcal{M}$. Then, on a manifold $\mathcal{M}$, we will have the induced metric: $h_{ij}(x) = Q^A_i(x)Q^B_j(x)G_{AB}(Q(x))$.

The stochastic process $\xi(t) = \{x^i(t)\}$, with the differential generator $\frac{1}{2}\triangle_{\mathcal{M}}$ ($\triangle_{\mathcal{M}}$ is a Laplace–Beltrami operator on $\mathcal{M}$), can be determined by the solution of the stochastic differential equation which has the following local representation:

$$\mathrm{d}x^k(t) = -\frac{1}{2}h^{ij}(x(t))\Gamma^k_{ij}(x(t))\,\mathrm{d}t + X^k_{\bar{m}}(x(t))\,\mathrm{d}w^{\bar{m}}(t) \qquad \left(\sum_{\bar{m}} X^k_{\bar{m}} X^l_{\bar{m}} = h^{kl}\right). \tag{A.1}$$

Now, we will define the same stochastic process, but instead of the variables x^i we will make use of the variables Q^A. We assume that the stochastic differential equation which describes the stochastic process on a submanifold can be written as

$$\mathrm{d}Q^A(t) = a^A\,\mathrm{d}t + \tilde{\mathfrak{X}}^A_{\bar{M}}\,\mathrm{d}w^{\bar{M}}(t) \tag{A.2}$$

where a^A and $\tilde{\mathfrak{X}}^A_{\bar{M}}(t)$ are some (and not yet defined) functions of $Q(t)$. Also, we assume that at the initial moment the process $Q^A(t)$ is given on a submanifold $\mathcal{M}$.

In order to find the explicit expressions for the coefficients a^A and $\tilde{\mathfrak{X}}^A_{\bar{M}}$ in equation (A.2) we will apply the Itô differentiation formula to the function $Q^A = Q^A(x^i(t))$. In the obtained expression, together with the ordinary derivatives will be the differentials of the stochastic variables $x^i(t)$. For these differentials the expressions can be taken from equation (A.1).

Then, comparing the result of such a differentiation with the expression on the right-hand side of (A.2), we find that the coefficient a^A is equal to

$$a^A = -\tfrac{1}{2}Q^A_i(x(t))h^{kl}(x(t))\Gamma^i_{kl}(x(t)) + \tfrac{1}{2}Q^A_{ij}(x(t))h^{ij}(x(t)). \tag{A.3}$$

But,

$$h^{kl}(x)\Gamma^i_{kl}(x) = G_{AB}(Q(x))\left(Q^A_{kl}(x) + \Gamma^A_{CD}(Q(x))Q^C_k(x)Q^D_l(x)\right)h^{im}(x)Q^B_m(x)h^{kl}(x) \tag{A.4}$$

(see, for example, [22]). Taking this into account and using the projection operators onto the tangent space to the manifold $\mathcal{M}$:

$$N^C_B(Q(x)) = G_{BA}(Q(x))Q^A_i(x)h^{ij}(x)Q^C_j(x)$$

from (A.3) we can obtain another expression for a^A:

$$a^A = -\tfrac{1}{2}N^A_P h^{ij}Q^C_i Q^D_j \Gamma^P_{CD} - \tfrac{1}{2}N^A_P Q^P_{kl}h^{kl} + \tfrac{1}{2}Q^A_{kl}h^{kl}. \tag{A.5}$$

Since the components of the mean curvature vector of the submanifold are given by

$$\begin{aligned} j^D &= \frac{1}{2}\left(\delta^D_B - N^D_B\right)h^{ij}\left[\nabla_{Q^P_i \frac{\partial}{\partial Q^P}}\left(Q^L_j \frac{\partial}{\partial Q^L}\right)\right]^B \\ &= \frac{1}{2}h^{ij}\left(Q^A_i Q^B_j \Gamma^D_{AB} + Q^D_{ij} - N^D_C Q^A_i Q^B_j \Gamma^C_{AB} - N^D_C Q^C_{ij}\right) \end{aligned}$$

we can rewrite (A.5) as follows:

$$a^A(Q(x)) = -\tfrac{1}{2}G^{EM}(Q(x))N^C_E(Q(x))N^B_M(Q(x))\Gamma^A_{CB}(Q(x)) + j^A \tag{A.6}$$

where j^A is, in fact, the function given on a submanifold, i.e., $j^A \equiv j^A(Q(x))$. This follows from the fact that the mean curvature can also be defined without using an explicit coordinate expression (for example, by the Weingarten map).

Before proceeding to the determination of the diffusion coefficient $\tilde{\mathfrak{X}}^A_{\bar{M}}(t)$, we note that the difussion coefficients of equations (A.1) and (A.2) are determined only up to the orthogonal transformations.

The Itô differentiation of $Q(x(t))$ also gives the equality:

$$\tilde{\mathfrak{X}}^A_{\bar{M}}\,\mathrm{d}w^{\bar{M}} = Q^A_i X^i_{\bar{m}}\,\mathrm{d}w^{\bar{m}}.$$

It then follows that

$$\sum_{\bar{M}} \tilde{\mathfrak{X}}^A_{\bar{M}} \tilde{\mathfrak{X}}^B_{\bar{M}} = \sum_{\bar{m}} Q^A_i X^i_{\bar{m}} Q^B_j X^j_{\bar{m}} = h^{ij} Q^A_i Q^B_j = G^{CD} N^A_C N^B_D.$$

Using these equations, we define $\tilde{\mathfrak{X}}^A_{\bar{M}}$ as

$$\tilde{\mathfrak{X}}^A_{\bar{M}} = N^A_C \mathfrak{X}^C_{\bar{M}} \qquad \left(\sum_{\bar{M}} \mathfrak{X}^D_{\bar{M}} \mathfrak{X}^C_{\bar{M}} = G^{CD} \right).$$

Finally, redefining the coordinates $Q^A(x(t))$ of the stochastic process for new coordinates $Q^A(t)$ (together with the requirement, that at the initial moment of time a new process also be given on a submanifold M), we get the following local stochastic differential equation for the components of the stochastic process on a submanifold $\mathcal{M}$:

$$\mathrm{d}Q^A(t) = \left(-\tfrac{1}{2} G^{EM} N^C_E N^D_M \Gamma^A_{CD} + j^A\right) \mathrm{d}t + N^A_C \mathfrak{X}^C_{\bar{M}} \, \mathrm{d}w^{\bar{M}} \tag{A.7}$$

where all the functions on the right-hand side of this equation now depend on $Q^A(t)$.

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